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1996 J. Phys. A: Math. Gen. 29 5719

(http://iopscience.iop.org/0305-4470/29/17/041)

J. Phys. A: Math. Gen. **29** (1996) 5719–5726. Printed in the UK

A quasiprobability based criterion for classifying the states of N spin-$\frac{1}{2}$s as classical or non-classical

R R Puri

Theoretical Physics Division, Central Complex, Bhabha Atomic Research Centre, Bombay-400085, India

Received 5 October 1995, in final form 9 April 1996

Abstract. Based on the concept of joint quasiprobability of the eigenstates of non-commuting components of a spin, a criterion is proposed to characterize the states of a system of N spin-$\frac{1}{2}$s as classical or non-classical. That criterion characterizes any state of a spin-$\frac{1}{2}$ and a factorizable state of N spin-$\frac{1}{2}$s as classical. It correctly identifies any pure entangled state of two spin-$\frac{1}{2}$s, an entangled eigenstate of any component of exchange symmetric spin system and squeezed spin states as non-classical.

The relationship between the quantum indeterminism and the classical statistical one has been of fundamental interest since the advent of quantum mechanics. In the case of a system of observables obeying the canonical commutation relations, that question is addressed by introducing the concept of phase space quasiprobability distributions (QPDs). That leads to the classification of the states as classical or non-classical. From amongst various quasiprobability distributions, the one that has proved most valuable for the said classification is the P-function: a state of a canonical system is labelled classical if its P-function is a classical distribution function. According to that criterion, the canonical coherent states are classical whereas all the other pure states of a canonical system are non-classical.

The non-classicality of a state of a system of spin-$\frac{1}{2}$s is characterized by that of spin–spin correlations [1–4]: a spin state is labelled non-classical if the spin–spin correlations cannot be described in terms of a classical distribution function. According to that scheme of classification, an uncorrelated state of spins, like a spin coherent state, as also all the states of a single spin-$\frac{1}{2}$, should therefore be identified as classical. However, the phase space distributions for spins [5–7], defined in analogy with those for a canonical system, do not provide any criterion for the said classification. That is because the phase space Wigner as well as the P-functions for spins turn out to be non-classical [5, 6]. For a system of two or more spin-$\frac{1}{2}$s the identification of non-classical states has been based largely on testing the Bell kind of inequalities or the GHZ equalities [1–3, 8]. Those tests, although they are of immense practical value, however, provide sufficient but not necessary conditions for a spin state to be non-classical. Here we introduce a criterion for classifying the states of a system of N spin-$\frac{1}{2}$s as classical or non-classical on the basis of the nature of spin–spin correlations. That criterion is based on characterizing any state of a spin-$\frac{1}{2}$ and an uncorrelated state of N spin-$\frac{1}{2}$s as classical. The ground and the highest excited eigenstates of any component of

0305-4470/96/175719+08\$19.50

the collective spin operator, i.e. the spin coherent states, being uncorrelated, are therefore classical, whereas all its other eigenstates which are entangled turn out to be non-classical. The classification of the states of a spin according to the criterion introduced here is, therefore, similar to that for a canonical system.

Our approach is based on the concept of joint quasiprobability for the eigenstates of two or more non-commuting spin components [8–13]. That quasiprobability is derived, following [13], by constructing first a classical analogue of the quantum system of N spin-$\frac{1}{2}$s described by the spin operators $\hat{\boldsymbol{S}}^{(i)}$, ($i = 1, 2, \ldots, N$) whose components $(\hat{S}_{\boldsymbol{a}}^{(i)}, \hat{S}_{\boldsymbol{b}}^{(i)}) \equiv (\hat{\boldsymbol{S}}^{(i)} \cdot \boldsymbol{a}, \hat{\boldsymbol{S}}^{(i)} \cdot \boldsymbol{b})$ along arbitarary directions $\boldsymbol{a}$ and $\boldsymbol{b}$ obey the anticommutation relation

$$\hat{S}_{\boldsymbol{a}}^{(i)} \hat{S}_{\boldsymbol{b}}^{(i)} + \hat{S}_{\boldsymbol{b}}^{(i)} \hat{S}_{\boldsymbol{a}}^{(i)} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{2} \tag{1}$$

and the commutation relation

$$[\hat{S}_{\boldsymbol{a}}^{(i)}, \hat{S}_{\boldsymbol{b}}^{(i)}] = \mathrm{i}(\boldsymbol{a} \times \boldsymbol{b}) \cdot \hat{\boldsymbol{S}}^{(i)}. \tag{2}$$

The operators for different spins, of course, commute. Quantum mechanically, the outcome of any measurement on an observable is an eigenvalue of that observable. Since the eigenvalues of any component of a spin-$\frac{1}{2}$ are $\pm\frac{1}{2}$, a measurement on a component of spin-$\frac{1}{2}$ results in one of the two values $\pm\frac{1}{2}$. The classical analogue of a system of N spin-$\frac{1}{2}$s is constructed accordingly by treating each of the operators $\hat{S}_{\boldsymbol{a}}^{(i)}$ as a classical two-state random variable $S_{\boldsymbol{a}}^{(i)}$ capable of assuming the values $\pm\frac{1}{2}$. The moments of those variables are determined in terms of the expectation values of the quantum operators. Those moments are then used to construct the classical probability distribution.

In what follows we need to work with the joint probability of three components, say, the components $S_{\boldsymbol{\mu}^{(i)}}^{(i)} \equiv \boldsymbol{S}^{i} \cdot \boldsymbol{\mu}^{(i)}$ along the directions $\boldsymbol{\mu}^{(i)} = \boldsymbol{a}^{(i)}, \boldsymbol{b}^{(i)}, \boldsymbol{c}^{(i)}$. Let $p_m^c(\epsilon_{\boldsymbol{a}^{(1)}}^{(1)}, \epsilon_{\boldsymbol{b}^{(1)}}^{(1)}, \epsilon_{\boldsymbol{c}^{(1)}}^{(1)}; \epsilon_{\boldsymbol{a}^{(2)}}^{(2)}, \epsilon_{\boldsymbol{b}^{(2)}}^{(2)}, \epsilon_{\boldsymbol{c}^{(2)}}^{(2)}; \ldots \epsilon_{\boldsymbol{a}^{(m)}}^{(m)}, \epsilon_{\boldsymbol{b}^{(m)}}^{(m)}, \epsilon_{\boldsymbol{c}^{(m)}}^{(m)})$, where $\epsilon_{\boldsymbol{\mu}^{(i)}}^{(i)} = \pm 1$ denotes the joint probability for those components of spins $1, 2, \ldots, m$ to have the values $(\epsilon_{\boldsymbol{a}^{(1)}}^{(1)}/2, \epsilon_{\boldsymbol{b}^{(1)}}^{(1)}/2, \epsilon_{\boldsymbol{c}^{(1)}}^{(1)}/2; \epsilon_{\boldsymbol{a}^{(2)}}^{(2)}/2, \epsilon_{\boldsymbol{b}^{(2)}}^{(2)}/2, \epsilon_{\boldsymbol{c}^{(2)}}^{(2)}/2; \ldots \epsilon_{\boldsymbol{a}^{(m)}}^{(m)}/2, \epsilon_{\boldsymbol{b}^{(m)}}^{(m)}/2, \epsilon_{\boldsymbol{c}^{(m)}}^{(m)}/2)$, respectively. The probability distribution for three components each of the m spins is then evidently given by

$$\begin{aligned} f_m^c(\{S_{\boldsymbol{a}^{(i)}}^{(i)}, S_{\boldsymbol{b}^{(i)}}^{(i)}, S_{\boldsymbol{c}^{(i)}}^{(i)}\}_m) = &\sum_{\{\epsilon_{\mu^{(j)}}^{(j)} = \pm 1\}} p_m^c(\{\epsilon_{\boldsymbol{a}^{(j)}}^{(j)}, \epsilon_{\boldsymbol{b}^{(j)}}^{(j)}, epsilon_{\boldsymbol{c}^{(j)}}^{(j)}\}_m) \\ &\times \prod_{i=1}^{m} \delta\left(S_{\boldsymbol{a}^{(i)}}^{(i)} - \frac{\epsilon_{\boldsymbol{a}^{(i)}}^{(i)}}{2}\right) \delta\left(S_{\boldsymbol{b}^{(i)}}^{(i)} - \frac{\epsilon_{\boldsymbol{b}^{(i)}}^{(i)}}{2}\right) \delta\left(S_{\boldsymbol{c}^{(i)}}^{(i)} - \frac{\epsilon_{\boldsymbol{c}^{(i)}}^{(i)}}{2}\right) \end{aligned} \tag{3}$$

where $(\{x^{(i)}, y^{(i)}, z^{(i)}\}_m) \equiv (x^{(1)}, y^{(1)}, z^{(1)}; x^{(2)}, y^{(2)}, z^{(2)}; \ldots x^{(m)}, y^{(m)}, z^{(m)})$. The expression for the joint probability derived in [13] in terms of the averages can be written in the following useful form:

$$p_m^c(\{\epsilon_{\boldsymbol{a}^{(i)}}^{(i)}, \epsilon_{\boldsymbol{b}^{(i)}}^{(i)}, \epsilon_{\boldsymbol{c}^{(i)}}^{(i)}\}_m) = \left\langle \prod_{j=1}^{m} (\tfrac{1}{2} + \epsilon_{\boldsymbol{a}^{(j)}}^{(j)} S_{\boldsymbol{a}^{(j)}}^{(j)})(\tfrac{1}{2} + \epsilon_{\boldsymbol{b}^{(j)}}^{(j)} S_{\boldsymbol{b}^{(j)}}^{(j)})(\tfrac{1}{2} + \epsilon_{\boldsymbol{c}^{(j)}}^{(j)} S_{\boldsymbol{c}^{(j)}}^{(j)}) \right\rangle \tag{4}$$

where the angular bracket denotes the average with respect to the given distribution. The expression (4) can alternatively be derived by multiplying the two sides of (3) by $\prod_{j=1}^{m}(\frac{1}{2} + \alpha_{\boldsymbol{a}^{(j)}}^{(j)} S_{\boldsymbol{a}^{(j)}}^{(j)})(\frac{1}{2} + \alpha_{\boldsymbol{b}^{(j)}}^{(j)} S_{\boldsymbol{b}^{(j)}}^{(j)})(\frac{1}{2} + \alpha_{\boldsymbol{c}^{(j)}}^{(j)} S_{\boldsymbol{c}^{(j)}}^{(j)})$, where $\alpha_{\boldsymbol{\mu}^{(j)}}^{(j)} = \pm 1$, followed by integration over the random variables to obtain

$$\left\langle \prod_{j=1}^{m} (\tfrac{1}{2} + \alpha_{\boldsymbol{a}^{(j)}}^{(j)} S_{\boldsymbol{a}^{(j)}}^{(j)})(\tfrac{1}{2} + \alpha_{\boldsymbol{b}^{(j)}}^{(j)} S_{\boldsymbol{b}^{(j)}}^{(j)})(\tfrac{1}{2} + \alpha_{\boldsymbol{c}^{(j)}}^{(j)} S_{\boldsymbol{c}^{(j)}}^{(j)}) \right\rangle = \sum_{\{\epsilon_{\mu^{(i)}}^{(i)} = \pm 1\}} p_m^c(\{\epsilon_{\boldsymbol{a}^{(i)}}^{(i)}, \epsilon_{\boldsymbol{b}^{(i)}}^{(i)}, \epsilon_{\boldsymbol{c}^{(i)}}^{(i)}\}_m)$$

$$\times \prod_{j=1}^{m} [\{\tfrac{1}{2}(1+\epsilon^{(j)}_{\boldsymbol{a}^{(j)}}\alpha^{(j)}_{\boldsymbol{a}^{(j)}})\}\{\tfrac{1}{2}(1+\epsilon^{(j)}_{\boldsymbol{b}^{(j)}}\alpha^{(j)}_{\boldsymbol{b}^{(j)}})\}\{\tfrac{1}{2}(1+\epsilon^{(j)}_{\boldsymbol{c}^{(j)}}\alpha^{(j)}_{\boldsymbol{c}^{(j)}})\}]. \tag{5}$$

The expression (4) follows by noting that the right-hand side of (5) is non-zero only when $\epsilon^{(j)}_{\boldsymbol{a}^{(j)}}\alpha^{(j)}_{\boldsymbol{a}^{(j)}} = \epsilon^{(j)}_{\boldsymbol{b}^{(j)}}\alpha^{(j)}_{\boldsymbol{b}^{(j)}} = \epsilon^{(j)}_{\boldsymbol{c}^{(j)}}\alpha^{(j)}_{\boldsymbol{c}^{(j)}} = 1$, i.e. when $\epsilon^{(j)}_{\boldsymbol{a}^{(j)}} = \alpha^{(j)}_{\boldsymbol{a}^{(j)}}, \epsilon^{(j)}_{\boldsymbol{b}^{(j)}} = \alpha^{(j)}_{\boldsymbol{b}^{(j)}}, \epsilon^{(j)}_{\boldsymbol{c}^{(j)}} = \alpha^{(j)}_{\boldsymbol{c}^{(j)}}$ for all j.

The quantum analogue p^q_m of the classical joint probability p^c_m, i.e. the joint quasiprobability (JQP), is constructed by reverting to the quantum description by replacing the classical random variables $S^{(i)}_{\boldsymbol{\mu}^{(i)}}$ by the spin operators $\hat{S}^{(i)}_{\boldsymbol{\mu}(i)}$. That procedure, however, encounters the well known conceptual problem of finding a quantum counterpart of the products of those classical variables whose quantum analogues are represented by non-commuting operators. The choice of different rules of association of the products of classical variables with those of the quantum operators leads to different quasiprobabilities. We choose the 'Wigner-like' symmetric ordering of the products whereby, also invoking (1),

$$\begin{aligned} S^{(i)}_{\boldsymbol{a}} S^{(i)}_{\boldsymbol{b}} &\to \frac{1}{2}(\hat{S}^{(i)}_{\boldsymbol{a}}\hat{S}^{(i)}_{\boldsymbol{b}} + \hat{S}^{(i)}_{\boldsymbol{b}}\hat{S}^{(i)}_{\boldsymbol{a}}) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{4} \text{ etc} \\ S^{(i)}_{\boldsymbol{a}} S^{(i)}_{\boldsymbol{b}} S^{(i)}_{\boldsymbol{c}} &\to \tfrac{1}{12}[\hat{S}^{(i)}_{\boldsymbol{a}}(\hat{S}^{(i)}_{\boldsymbol{b}}\hat{S}^{(i)}_{\boldsymbol{c}} + \hat{S}^{(i)}_{\boldsymbol{c}}\hat{S}^{(i)}_{\boldsymbol{b}}) + (\hat{S}^{(i)}_{\boldsymbol{b}}\hat{S}^{(i)}_{\boldsymbol{c}} + \hat{S}^{(i)}_{\boldsymbol{c}}\hat{S}^{(i)}_{\boldsymbol{b}})\hat{S}^{(i)}_{\boldsymbol{a}} \\ &\quad + (a\to b, b\to c, c\to a) + (a\to c, c\to b, b\to a)] \\ &= \tfrac{1}{6}[(\boldsymbol{b}\cdot\boldsymbol{c})\hat{S}^{(i)}_{\boldsymbol{a}} + (\boldsymbol{a}\cdot\boldsymbol{c})\hat{S}^{(i)}_{\boldsymbol{b}} + (\boldsymbol{a}\cdot\boldsymbol{b})\hat{S}^{(i)}_{\boldsymbol{c}}]. \end{aligned} \tag{6}$$

The merit of the symmetric ordering will become clear in what follows. The JQP can be constructed by replacing the classical variables in (4) by the corresponding operators using the rule (6) and by interpreting the averages as the expectation values with respect to the density matrix describing the given quantum state. The expression for the JQP simplifies considerably for mutually orthogonal components. For if $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are mutually orthogonal then the correspondence (6) leads to

$$\left(\tfrac{1}{2} + S_a\right)\left(\tfrac{1}{2} + S_b\right)\left(\tfrac{1}{2} + S_c\right) \to \frac{1}{2^2}\left(\tfrac{1}{2} + \hat{S}_{\boldsymbol{a}} + \hat{S}_{\boldsymbol{b}} + \hat{S}_{\boldsymbol{c}}\right). \tag{7}$$

On using (7) in (4) the expression for the JQP for three mutually orthogonal components of each of the spins reads

$$p^q_m(\{\epsilon^{(i)}_{\boldsymbol{a}_i}, \epsilon^{(i)}_{\boldsymbol{b}_i}, \epsilon^{(i)}_{\boldsymbol{c}_i}\}_m) = \left\langle \prod_{j=1}^{m} \frac{1}{2^2}\left(\frac{1}{2} + \epsilon^{(j)}_{\boldsymbol{a}_j}\hat{S}^{(j)}_{\boldsymbol{a}_j} + \epsilon^{(j)}_{\boldsymbol{b}_j}\hat{S}^{(j)}_{\boldsymbol{b}_j} + \epsilon^{(j)}_{\boldsymbol{c}_j}\hat{S}^{(j)}_{\boldsymbol{c}_j}\right)\right\rangle. \tag{8}$$

In what follows we will need to deal with the JQP for three mutually orthogonal components of each of the spins.

Before proceeding further let us point out the connection between the JQP and the phase space distributions for the spins. To that end we show that the joint probability for finding each spin in the eigenstate $|(\frac{1}{2})_i, \boldsymbol{a}\rangle(|(-\frac{1}{2})_i, \boldsymbol{a}\rangle)$ of its component in the direction $\boldsymbol{a}$ is the Q-function $\langle\boldsymbol{a}|\rho|\boldsymbol{a}\rangle(\langle-\boldsymbol{a}|\rho|-\boldsymbol{a}\rangle)$ where ρ is the density matrix of the system of spins and $|\pm\boldsymbol{a}\rangle$ are the spin coherent states. Recall that the spin coherent state $|\boldsymbol{a}\rangle$ $(|-\boldsymbol{a}\rangle)$ is the product of single-spin eigenstates $|(\frac{1}{2})_i, \boldsymbol{a}\rangle$ $(|(-\frac{1}{2})_i, \boldsymbol{a}\rangle)$:

$$|\pm\boldsymbol{a}\rangle = \prod_{i=1}^{N} |\pm(\tfrac{1}{2})_i, \boldsymbol{a}\rangle \tag{9}$$

where

$$\hat{S}^{(i)}_{\boldsymbol{a}}|\pm(\tfrac{1}{2})_i, \boldsymbol{a}\rangle = \pm\tfrac{1}{2}|\pm(\tfrac{1}{2})_i, \boldsymbol{a}\rangle. \tag{10}$$

Using (9) and the identity

$$|\pm(\tfrac{1}{2})_i, \boldsymbol{a}\rangle\langle\pm(\tfrac{1}{2})_i, \boldsymbol{a}| = (\tfrac{1}{2} \pm \hat{S}_{\boldsymbol{a}}^{(i)}) \tag{11}$$

the Q-function $\langle\pm\boldsymbol{a}|\rho|\pm\boldsymbol{a}\rangle$ can obviously be written as

$$\begin{aligned}\langle\pm\boldsymbol{a}|\rho|\pm\boldsymbol{a}\rangle &= \mathrm{Tr}[\rho|\pm\boldsymbol{a}\rangle\langle\pm\boldsymbol{a}|] \\ &= \left\langle \prod_{i=1}^{N}(\tfrac{1}{2} \pm \hat{S}_{\boldsymbol{a}}^{(i)}) \right\rangle\end{aligned} \tag{12}$$

where the angular brackets denote expectation value with respect to the density matrix ρ. On the other hand, the classical expression for the joint probability for the component of each of the spins in the direction $\boldsymbol{a}$ to have the value $\pm\frac{1}{2}$ is given, invoking (4), by

$$p_N^c(\{\pm\}_N) = \left\langle \prod_{j=1}^{N}(\tfrac{1}{2} \pm S_{\boldsymbol{a}}^{(j)}) \right\rangle. \tag{13}$$

The quantum analogue of the joint probability obtained by replacing the classical variables in (13) by the operators reads

$$p_N^q(\{\pm\}_N) = \left\langle \prod_{j=1}^{N}(\tfrac{1}{2} \pm \hat{S}_{\boldsymbol{a}}^{(j)}) \right\rangle \tag{14}$$

which is the same as the expression (12) for the Q-function. The joint probability (14) is always classical since it refers to the measurement of only commuting components.

The non-classical behaviour can be reflected in the joint probability involving more than one component of each of the spins. In order to choose the components, the joint probability distribution of whose eigenstates is to be used to classify the spin states as classical or non-classical, it is essential to specify the property on which one wishes to base that classification. As discussed above, the property of interest for a system of spins is the spin–spin correlations [1–4]: a spin state is labelled classical or non-classical according to whether the spin–spin correlations are classical or not. That implies that any criterion used to classify the states of a system of N spin-$\frac{1}{2}$s should identify any state of a single spin-$\frac{1}{2}$ and also an uncorrelated state of spins as classical. Now, any pure state of a spin-$\frac{1}{2}$ is an eigenstate of some spin component. Consider then an eigenstate $|\pm\boldsymbol{a}\rangle$ of the component $\hat{S}_{\boldsymbol{a}}^{(i)}$ of the ith spin-$\frac{1}{2}$. For such a state

$$\langle\pm\boldsymbol{a}|\hat{S}_{\boldsymbol{a}}^{(i)}|\pm\boldsymbol{a}\rangle = \pm\tfrac{1}{2} \qquad \langle\pm\boldsymbol{a}|\hat{S}_{\boldsymbol{\mu}}^{(i)}|\pm\boldsymbol{a}\rangle = 0 \qquad \text{where } \boldsymbol{\mu}\cdot\boldsymbol{a} = 0 \tag{15}$$

i.e. the average direction of the spin in state $|\pm\boldsymbol{a}\rangle$ is $\pm\boldsymbol{a}$. Also, the components of the spin orthogonal to that direction and to each other are uncorrelated, i.e.

$$\langle\hat{S}_{\boldsymbol{\mu}}^{(i)}\hat{S}_{\boldsymbol{\nu}}^{(i)}\rangle + \langle\hat{S}_{\boldsymbol{\nu}}^{(i)}\hat{S}_{\boldsymbol{\mu}}^{(i)}\rangle - 2\langle\hat{S}_{\boldsymbol{\mu}}^{(i)}\rangle\langle\hat{S}_{\boldsymbol{\nu}}^{(i)}\rangle = 0 \qquad \boldsymbol{\mu} \neq \boldsymbol{\nu} = \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \tag{16}$$

where $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are orthogonal to each other. Now, the analogous classical system is constructed, as discussed before, by demanding that the averages of the classical random variables $S_{\boldsymbol{\mu}}^{(i)}$ corresponding to the operators $\hat{S}_{\boldsymbol{\mu}}^{(i)}$ be given by (15). As a result, the corresponding classical correlation function

$$C_{(\boldsymbol{\mu},\boldsymbol{\nu})}^{(ii)} = \langle S_{\boldsymbol{\mu}}^{(i)} S_{\boldsymbol{\nu}}^{(i)}\rangle - \langle S_{\boldsymbol{\mu}}^{(i)}\rangle\langle S_{\boldsymbol{\nu}}^{(i)}\rangle \qquad \boldsymbol{\mu} \neq \boldsymbol{\nu} = \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \tag{17}$$

reduces to $\langle S_{\boldsymbol{\mu}}^{(i)} S_{\boldsymbol{\nu}}^{(i)}\rangle$. That function would vanish, like its quantum counterpart (16), if the classical product is identified with the symmetric product of the operators as in (6). In other orderings, for example the ones discussed in [11, 12], the correlation function (17) will be non-zero for some $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. Thus the Wigner-like symmetric ordering presents itself as a natural choice if the classical variables corresponding to the uncorrelated quantum operators are also to be uncorrelated. For mutually orthogonal vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ the JQP is given by (8). For a single spin-$\frac{1}{2}$ it reads

$$p^q(\epsilon_{\boldsymbol{a}}^{(i)}, \epsilon_{\boldsymbol{b}}^{(i)}, \epsilon_{\boldsymbol{c}}^{(i)}) = \frac{1}{2^3}(1 \pm \epsilon_{\boldsymbol{a}}^{(i)}). \tag{18}$$

The JQP (18) is always positive semi-definite, i.e. classical. It can be verified that the JQP for any other choice of spin components will not be classical.

Next we consider an uncorrelated system of N spin-$\frac{1}{2}$s. The JQP for such a system evidently factorizes. Consider the joint quasiprobabilities $p_m(\{\epsilon_{\boldsymbol{a}^{(i)}}^{(i)}, \epsilon_{\boldsymbol{b}^{(i)}}^{(i)}, \epsilon_{\boldsymbol{c}^{(i)}}^{(i)}\}_m)$ $(m = 1, 2, \ldots, N)$ where $(\boldsymbol{a}^{(i)}, \boldsymbol{b}^{(i)}, \boldsymbol{c}^{(i)})$ are mutually orthogonal with $\boldsymbol{a}^{(i)}$ in the average direction of the ith spin. By noting that the average of the product of uncorrelated spins is the product of their averages, the expression (8) for the JQP reduces to

$$p^q(\{\epsilon_{\boldsymbol{a}}^{(i)}, \epsilon_{\boldsymbol{b}}^{(i)}\epsilon_{\boldsymbol{c}}^{(i)}\}_N) = \prod_{j=1}^{N} \frac{1}{2^3}(1 \pm \epsilon_{\boldsymbol{a}^{(j)}}^{(j)}). \tag{19}$$

Thus all the quasiprobabilities for the joint distribution of three orthogonal components of each of the spins are positive semi-definite, i.e. classical if one of those components for a spin is in the average direction of that spin. In particular, if the spins have exchange symmetry then the average direction of all the spins is the same; say $\pm\boldsymbol{a}$. The uncorrelated state of such a system is the spin coherent state. From (19) it follows that if the system is in the spin coherent state $|\frac{1}{2}, \boldsymbol{a}\rangle$ then

$$\begin{aligned} p_N^q(\{+, \epsilon_{\boldsymbol{b}}^{(i)}, \epsilon_{\boldsymbol{c}}^{(i)}\}_N) &= \frac{1}{2^{2N}} \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{20}$$

whereas for the state $|-\frac{1}{2}, \boldsymbol{a}\rangle$, $p_N^q(\{-, \epsilon_{\boldsymbol{b}}^{(i)}, \epsilon_{\boldsymbol{c}}^{(i)}\}_N) = 1/2^{2N}$ are the only non-zero JQPs. The corresponding quasiprobability distribution, obtained by substituting (20) in the quantum version of (3), reads

$$\begin{aligned} f_N^q(\{S_{\boldsymbol{a}}, S_{\boldsymbol{b}}, S_{\boldsymbol{c}}\}_N) &= \frac{1}{2^{2N}} \sum_{\{\epsilon_{\mu=\pm 1}^{(i)}\}} \prod_{i=1}^{N} \delta\left(S_{\boldsymbol{a}}^{(i)} \pm \frac{1}{2}\right)\delta\left(S_{\boldsymbol{b}}^{(i)} - \frac{\epsilon_{\boldsymbol{b}^{(i)}}^{(i)}}{2}\right)\delta\left(S_{\boldsymbol{c}}^{(i)} - \frac{\epsilon_{\boldsymbol{c}^{(i)}}^{(i)}}{2}\right) \\ &\equiv \frac{1}{2^{2N}} \prod_{i=1}^{N} [\delta(S_{\boldsymbol{a}}^{(i)} \pm \tfrac{1}{2})\{\delta(S_{\boldsymbol{b}}^{(i)} - \tfrac{1}{2}) + \delta(S_{\boldsymbol{b}}^{(i)} + \tfrac{1}{2})\}\{\delta(S_{\boldsymbol{c}}^{(i)} - \tfrac{1}{2}) + \delta(S_{\boldsymbol{c}}^{(i)} + \tfrac{1}{2})\}]. \end{aligned} \tag{21}$$

The quasiprobability distribution of the chosen non-commuting components factorizes which implies that those components behave like independent classical random variables. We therefore propose the following criterion for classifying the states of N spin-$\frac{1}{2}$s.

A given quantum state of a system of N spin-$\frac{1}{2}$s is classical if the joint quasiprobability for the eigenstates of the components of each spin in three mutually orthogonal directions, one of which is the average direction of that spin, is classical in the symmetric ordering of the operators. The given quantum state is non-classical if any of those m-spin ($m \leqslant N$) joint quasiprobabilities is negative in that ordering.

It is straightforward to see that not only all the spin-$\frac{1}{2}$ pure states but also any mixed spin-$\frac{1}{2}$ state is classical. Hence it follows that a spin state factorizable into single spin states is also classical. Let us therefore examine the nature of entangled states of a system of

N spin-$\frac{1}{2}$s. It can be verified that a positive JQP for the above-mentioned cases cannot be obtained for any other choice of spin components.

First we examine the nature of pure states of a system of two spin-$\frac{1}{2}$s. Any such state can be written, by Schmidt decomposition, in the form [14, 15]

$$|\psi, \alpha\rangle = \cos(\alpha)|+a_1, -a_2\rangle + \sin(\alpha)|-a_1, +a_2\rangle \tag{22}$$

where $|\pm a_i\rangle$ are the eigenstates of the component $\hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{a}_i}$ of the ith spin in the direction $\boldsymbol{a}_i$ corresponding to the eigenvalues $\pm\frac{1}{2}$. The parameter α is a measure of the degree of correlation between the two states. For $\alpha = 0$ the state is a product state, i.e. the spins in that state are uncorrelated whereas the maximal correlation is obtained for $\alpha = \pi/4$. In order to apply our criterion, we first introduce the spin operators $\hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{b}_i}, \hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{c}_i}$ in the two orthogonal directions $\boldsymbol{b}_i, \boldsymbol{c}_i$ which are orthogonal also to $\boldsymbol{a}_i$. The spin components so constructed obey the angular momentum commutation relations. Clearly the operators $\hat{\boldsymbol{S}}^{(i)}_{\pm} = \hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{b}_i} \pm \mathrm{i}\hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{c}_i}$ are the raising/lowering operators on the eigenstates $|\pm a_i\rangle$ of $\hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{a}_i}$. It is then straightforward to see that$\langle\hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{b}_i}\rangle = \langle\hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{c}_i}\rangle = 0$. Hence $\boldsymbol{a}_i$ is the average direction of the ith spin. The criterion for identifying the nature of the given state can therefore be applied by determining the JQP for the eigenstates of $\hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{a}_i}, \hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{b}_i}, \hat{\boldsymbol{S}}^{(i)}_{\boldsymbol{c}_i}$. The expression (8) for the JQP in that case can be shown to be given by

$$\begin{aligned} p_2^q(\{\epsilon^{(i)}_{\boldsymbol{a}_i}, \epsilon^{(i)}_{\boldsymbol{b}_i}, \epsilon^{(i)}_{\boldsymbol{c}_i}\}_m) = \frac{1}{2^6}[1 &+ \cos(2\alpha)(\epsilon^{(1)}_{\boldsymbol{a}_1} - \epsilon^{(2)}_{\boldsymbol{a}_2}) - \epsilon^{(1)}_{\boldsymbol{a}_1}\epsilon^{(2)}_{\boldsymbol{a}_2} \\ &+ \sin(2\alpha)(\epsilon^{(1)}_{\boldsymbol{b}_1}\epsilon^{(2)}_{\boldsymbol{b}_2} + \epsilon^{(1)}_{\boldsymbol{c}_1}\epsilon^{(2)}_{\boldsymbol{c}_2})]. \end{aligned} \tag{23}$$

If $\epsilon^{(1)}_{\boldsymbol{a}_1} = \epsilon^{(2)}_{\boldsymbol{a}_2} = \epsilon$ then

$$p_2^q(\epsilon, \epsilon^{(1)}_{\boldsymbol{b}_1}, \epsilon^{(1)}_{\boldsymbol{c}_1}; \epsilon, \epsilon^{(2)}_{\boldsymbol{b}_2}, \epsilon^{(2)}_{\boldsymbol{c}_2}) = \frac{1}{2^6}\sin(2\alpha)[\epsilon^{(1)}_{\boldsymbol{b}_1}\epsilon^{(2)}_{\boldsymbol{b}_2} + \epsilon^{(1)}_{\boldsymbol{c}_1}\epsilon^{(2)}_{\boldsymbol{c}_2}]. \tag{24}$$

The JQP (24) can clearly become negative if $\alpha \neq 0$, i.e. if the states are entangled. If $\alpha = 0$, i.e. when the states are uncorrelated, then the JQP factorizes and is classical. Thus it follows that any pure entangled state of two spin-$\frac{1}{2}$s is non-classical. This is in agreement with the results of [14, 15] where the non-classicality of a pure entangled state of two spin-$\frac{1}{2}$s is established by showing that those states violate Bell's inequality. Note that a state which is non-classical in the sense of the criterion introduced here need not violate Bell's inequality. Bell's inequality is a sufficient but not a necessary condition for a state to be non-classical.

As for the mixed entangled state of two spin-$\frac{1}{2}$s, it has been shown in [16] that a state of two spin-$\frac{1}{2}$s which is a sum of its factorizable states obeys Bell's inequality. However, as has already been mentioned, a state obeying Bell's inequality need not be classical in the sense of our criterion.

We now examine the nature of the states of a system of N spin-$\frac{1}{2}$s for $N > 2$. We restrict our attention to a collective system of N spin-$\frac{1}{2}$s having total spin $S = N/2$. Such a system is symmetric under the exchange of spins. Consider first the eigenstates of a collective spin component $\hat{S}_{\boldsymbol{a}} = \sum_i \hat{S}^{(i)}_{\boldsymbol{a}}$ denoted by $|m, \boldsymbol{a}\rangle$ where $m = -N/2, -N/2+1, \ldots, N/2$. Now the states $|\pm N/2, \boldsymbol{a}\rangle$ are the spin coherent states which have already been shown to be classical. We will show that the states $|m \neq \pm N/2, \boldsymbol{a}\rangle$ are non-classical. Since the average direction of all the spins in those states is $\boldsymbol{a}$ we establish their non-classicality in accordance with the proposed criterion by showing that the JQP for the eigenstates of the components of two spins, say, spins i and j along mutually orthogonal directions $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, i.e.

$$\begin{aligned} p_2^q(\epsilon^{(i)}_{\boldsymbol{a}}, \epsilon^{(i)}_{\boldsymbol{b}}, \epsilon^{(i)}_{\boldsymbol{c}}; \epsilon^{(j)}_{\boldsymbol{a}}, \epsilon^{(j)}_{\boldsymbol{b}}, \epsilon^{(j)}_{\boldsymbol{c}}) = \frac{1}{2^4}\langle[\tfrac{1}{2} &+ \epsilon^{(i)}_{\boldsymbol{a}}\hat{S}^{(i)}_{\boldsymbol{a}} + \epsilon^{(i)}_{\boldsymbol{b}}\hat{S}^{(i)}_{\boldsymbol{b}} + \epsilon^{(i)}_{\boldsymbol{c}}\hat{S}^{(i)}_{\boldsymbol{c}})] \\ \times[\tfrac{1}{2} &+ \epsilon^{(j)}_{\boldsymbol{a}}\hat{S}^{(j)}_{\boldsymbol{a}} + \epsilon^{(j)}_{\boldsymbol{b}}\hat{S}^{(j)}_{\boldsymbol{b}} + \epsilon^{(j)}_{\boldsymbol{c}}\hat{S}^{(j)}_{\boldsymbol{c}})]\rangle \end{aligned} \tag{25}$$

is non-classical. Note that the state $|m \neq \pm N/2, \boldsymbol{a}\rangle$ does not factorize into the single atom states which implies that the spins in that state are correlated. For the evaluation of two-spin JQP (25) we need to know the expectation values of the products of the operators for two spins in the collective spin state $|m, \boldsymbol{a}\rangle$. Those expectation values can be determined by invoking the fact that the state $|m, \boldsymbol{a}\rangle$ is symmetric under the exchange of spins so that

$$\langle \hat{S}_\mu^{(i)} \hat{S}_\nu^{(j)} \rangle = \frac{1}{N(N-1)} \left[\langle \hat{S}_\mu \hat{S}_\nu \rangle - \sum_{i=1}^{N} \langle \hat{S}_\mu^{(i)} \hat{S}_\nu^{(i)} \rangle \right]. \tag{26}$$

Since the product of two single spin operators is expressible in terms of a single spin operator, the operators on the right-hand side of (26) are collective operators whose expectation values can be evaluated in the collective atomic states. Evaluating now the averages in (25) using (26) we get

$$p_2^q(\epsilon_{\boldsymbol{a}}^{(i)}, \epsilon_{\boldsymbol{b}}^{(i)}, \epsilon_{\boldsymbol{c}}^{(i)}; \epsilon_{\boldsymbol{a}}^{(j)}, \epsilon_{\boldsymbol{b}}^{(j)}, \epsilon_{\boldsymbol{c}}^{(j)}) = \frac{1}{64} \left[1 + \frac{2m}{N} (\epsilon_{\boldsymbol{a}}^{(i)} + \epsilon_{\boldsymbol{a}}^{(j)}) \right.$$
$$\left. + \frac{2}{N(N-1)} \left\{ (\epsilon_{\boldsymbol{b}}^{(i)} \epsilon_{\boldsymbol{b}}^{(j)} + \epsilon_{\boldsymbol{c}}^{(i)} \epsilon_{\boldsymbol{c}}^{(j)}) \left(\frac{N^2}{4} - m^2 \right) + 2\epsilon_{\boldsymbol{a}}^{(i)} \epsilon_{\boldsymbol{a}}^{(j)} \left(m^2 - \frac{N}{4} \right) \right\} \right]. \tag{27}$$

If $m \geqslant 0$ then let $\epsilon_{\boldsymbol{a}}^{(i)} = \epsilon_{\boldsymbol{a}}^{(j)} = \epsilon_{\boldsymbol{b}}^{(i)} \epsilon_{\boldsymbol{b}}^{(j)} = \epsilon_{\boldsymbol{c}}^{(i)} \epsilon_{\boldsymbol{c}}^{(j)} = -1$ so that

$$p_2^q(-1, \epsilon_2, \epsilon_3; -1, -\epsilon_2, -\epsilon_3) = -\tfrac{1}{32}(2m+1)(N-2m). \tag{28}$$

Since m is always less than or equal to $N/2$ and is assumed to be positive semidefinite it follows that the JQP (28) is negative. Similarly the JQP for $m \leqslant 0$ becomes negative if $\epsilon_{\boldsymbol{a}}^{(i)} = \epsilon_{\boldsymbol{a}}^{(j)} = 1$ with other ϵ's chosen in the same way as for $m \geqslant 0$. Hence the states $|m \neq \pm N/2, \boldsymbol{a}\rangle$ are non-classical.

We have thus proved that a symmetric entangled state of a system of N spin-$\frac{1}{2}$s is non-classical if it is an eigenstate of a linear Hermitian spin operator. An entangled state, however, need not be an eigenstate of a linear Hermitian spin operator. For example, the entangled minimum uncertainty states of spins [17] are eigenstates of the linear non-Hermitian spin operator $\cosh(\theta) S_x + \mathrm{i} \sinh(\theta) S_y$. Now we examine the nature of a symmetric entangled state of spins which is not necessarily an eigenstate of a Hermitian spin operator. Since the system is symmetric, the average direction of all the spins is the same. Let $\boldsymbol{a}$ be the average direction of the spins and let $\hat{\boldsymbol{S}}_{\boldsymbol{a}}^{(i)}$ be the component of the ith spin in that direction. Let $\boldsymbol{b}, \boldsymbol{c}$ be the directions orthogonal to each other and to $\boldsymbol{a}$ and let $\hat{\boldsymbol{S}}_{\boldsymbol{b}}^{(i)}, \hat{\boldsymbol{S}}_{\boldsymbol{c}}^{(i)}$ be the corresponding spin components. Since the spin vector is given by

$$\hat{\boldsymbol{S}} = \hat{\boldsymbol{S}}_a \boldsymbol{a} + \hat{\boldsymbol{S}}_b \boldsymbol{b} + \hat{\boldsymbol{S}}_c \boldsymbol{c} \tag{29}$$

and we have taken $\boldsymbol{a}$ to be in the direction of $\langle \hat{\boldsymbol{S}} \rangle$ it follows that

$$\langle \hat{\boldsymbol{S}}_{\boldsymbol{b}} \rangle = \langle \hat{\boldsymbol{S}}_{\boldsymbol{c}} \rangle = 0. \tag{30}$$

Consider now the JQP for two spins given by (8) for $m = 2$ corresponding to $\epsilon_{\boldsymbol{a}}^{(1)} = -\epsilon_{\boldsymbol{a}}^{(2)}$, $\epsilon_{\boldsymbol{b}}^{(1)} = -\epsilon_{\boldsymbol{b}}^{(2)}$, $\epsilon_{\boldsymbol{c}}^{(1)} = -\epsilon_{\boldsymbol{c}}^{(2)}$. By using (26), (30) and the fact that the eigenvalue of the total spin operator

$$\hat{S}^2 = \hat{S}_{\boldsymbol{a}}^2 + \hat{S}_{\boldsymbol{b}}^2 + \hat{S}_{\boldsymbol{c}}^2 \tag{31}$$

is $N(N+2)/4$ it follows that

$$p_2^q(\epsilon_{\boldsymbol{a}}^{(i)}, \epsilon_{\boldsymbol{b}}^{(i)}, \epsilon_{\boldsymbol{c}}^{(i)}; -\epsilon_{\boldsymbol{a}}^{(i)}, -\epsilon_{\boldsymbol{b}}^{(i)}, \epsilon_{\boldsymbol{c}}^{(i)})$$
$$= \frac{-1}{32N(N-1)} [\epsilon_{\boldsymbol{b}}^{(i)} \epsilon_{\boldsymbol{c}}^{(i)} \{ \langle \hat{S}_{\boldsymbol{b}} \hat{S}_{\boldsymbol{c}} \rangle - \langle S_{\boldsymbol{a}} \rangle \} + \epsilon_{\boldsymbol{a}}^{(i)} \epsilon_{\boldsymbol{b}}^{(i)} \langle \hat{S}_{\boldsymbol{a}} \hat{S}_{\boldsymbol{b}} \rangle + \epsilon_{\boldsymbol{a}}^{(i)} \epsilon_{\boldsymbol{c}}^{(i)} \langle \hat{S}_{\boldsymbol{a}} \hat{S}_{\boldsymbol{c}} \rangle]. \tag{32}$$

It is clear that depending upon the relative magnitudes of the averages appearing on the right-hand side of (32) there always exist ϵ's which make p_2^q negative. The only case for which it is non-negative is when all the averages in (32) vanish. Those averages vanish, of course, for a factorizable state. In that case the probabilities even for other values of ϵ's turn out to be positive; as they should. The averages in (32) are non-zero for squeezed spin states [17]. The squeezed spin states are, therefore, non-classical.

In conclusion, we have found a positive joint quasiprobability for any uncorrelated state of a system of N spin-$\frac{1}{2}$s. That JQP is shown to be negative for any entangled state of a system of two spin-$\frac{1}{2}$s. The nature of the states of a system of N spins symmetric under the exchange of spins has been examined. The entangled states of such a system which are eigenstates of a Hermitian linear spin operator have been shown to be non-classical. The condition for an entangled state which is not an eigenstate of a linear Hermitian operator to be classical has been obtained. In particular, the squeezed spin states, which are not eigenstates of a linear Hermitian operator, turn out to be non-classical. The classification of the spin states based on the criterion introduced here is similar to that for the canonical system.

Finally it should be emphasized that the criterion for the classification of the spin states, like the one based on the P-function for canonical operators, only identifies non-classical states. The problem of finding the experimentally observable quantities which carrry the signature of non-classicality is an altogether different issue.

Acknowledgments

The author is grateful to Professor G S Agarwal for useful discussions and to one of the referees for useful suggestions.

References

[1] Bell J S 1965 *Physics* **1** 195; 1966 *Rev. Mod. Phys.* **38** 447

[2] Greenberg D M, Horne M and Zeilinger A 1989 *Bell's Theorem, Quantum Theory and the Concepts of Universe* ed M Kafatos (Dordrecht: Kluwer)
Greenberg D M, Horne M, Shimony A and Zeilinger A 1990 *Am. J. Phys.* **58** 1131

[3] Mermin N D 1990 *Am. J. Phys.* **58** 731; 1990 *Phys. Today* **43** 9

[4] Kitagawa M and Ueda M 1993 *Phys. Rev.* A **47** 5138

[5] Agarwal G S 1983 *Phys. Rev.* A **24** 2889; 1993 *Phys. Rev.* A **47** 4608
Dowling J P, Agarwal G S and Schleich W P 1994 *Phys. Rev.* **49** 4101

[6] Wodkiewicz K 1988 *Phys. Lett.* **129A** 1
Scully M O and Wodkiewicz K 1990 *Coherence and Quantum Optics* ed J H Eberly, L Mandel and E Wolf (New York: Plenum) p 1047

[7] Takahashi Y and Shibata F 1975 *J. Phys. Soc. Japan* **38** 656; 1976 *J. Stat. Phys.* **14** 49

[8] Wigner E P 1981 Interpretation of quantum mechanics *Quantum Theory and Measurement* ed A J Wheeler and W H Zureck (Princeton, NJ: Princeton University Press)

[9] Feynman R P 1987 *Quantum Implications* ed B Hiley and F Peat (London: Routledge)

[10] Scully M O 1983 *Phys. Rev.* D **28** 2477

[11] Cohen L and Scully M O 1986 *Found. Phys.* **16** 295
Chandler C, Cohen L, Lu C, Scully M O and Wodkiewicz K 1992 *Found. Phys.* **22** 867

[12] Scully M O, Walther H and Schleich W 1994 *Phys. Rev.* A **49** 1562

[13] Puri R R 1995 *J. Phys. A: Math. Gen.* **28** 6227

[14] Gisin N 1991 *Phys. Lett.* **154A** 201

[15] Schumacher B W 1991 *Phys. Rev.* A **44** 7047

[16] Werner R F 1989 *Phys. Rev.* A **40** 4277

[17] Agarwal G S and Puri R R 1989 *Opt. Commun.* **69** 267; 1990 *Phys. Rev.* A **41** 3782; 1994 *Phys. Rev.* A **49** 4968; 1994 *Coherent States: Past Present and Future* ed J R Klauder and D H Feng (Singapore: World Scientific) p 447